Journal of Engineering Mathematics **25**: 191–205, 1991.

Integral results for nonlinear diffusion equations

J.R. KING

Department of Theoretical Mechanics, University of Nottingham, Nottingham, NG7 2RD, United Kingdom

Received 6 April 1990; accepted 7 June 1990

Abstract. We show how to construct integral results for the multi-dimensional nonlinear diffusion equation $\partial c/\partial t = \nabla\cdot(D(c)\nabla c)$, and for some generalisations of this. For appropriate boundary conditions these become integral invariants. An application of these results to determining the large-time behaviour of some radially symmetric problems is indicated.

1. Introduction

This paper is primarily concerned with nonlinear diffusion equations of the form

$$\frac{\partial c}{\partial t} = \nabla\cdot(D(c)\nabla c)\,, \tag{1.1}$$

which have a very large number of physical applications (see, for example, Crank [7]). Equation (1.1) may be rewritten in a more convenient form by introducing the Kirchhoff variable

$$w = \int_{c_0}^{c} D(c)\,\mathrm{d}c\,, \tag{1.2}$$

where c_0 is a specified constant; $c_0 = 0$ is usually chosen but that choice is clearly not acceptable when, for example, $D(c) \sim c^{-m}$ as $c \to 0^+$ with $m \geqslant 1$. Equation (1.1) may then be written as

$$\frac{\partial c}{\partial t} = \nabla^2 w\,. \tag{1.3}$$

In this paper we shall show in Section 2 how integral results for (1.3) may be determined by solving a Laplace equation. For appropriate boundary conditions these results then give integral invariants. In Section 3 we extend these results to an inhomogeneous version of (1.1) and in Section 4 we consider some higher-order nonlinear diffusion problems. In Section 5 we consider a simple radially symmetric problem for (1.1) which indicates how our results may be applied in, for example, determining the appropriate large-time behaviour. We conclude with some discussion.

2. Integral results for homogeneous diffusion

We start by obtaining integral results for (1.3) on a fixed, simply connected domain Ω with boundary $\partial\Omega$ by considering integrals of the form

$$I[f]=\int_{\Omega} f(\mathbf{x})c\,\mathrm{d}V\,, \tag{2.1}$$

where appropriate forms of $f(\mathbf{x})$ will be determined. We then have by (1.3)

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\int_{\Omega} f\nabla^2 w\,\mathrm{d}V\,,$$

so that

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\int_{\partial\Omega}\left(f\,\frac{\partial w}{\partial n}-w\,\frac{\partial f}{\partial n}\right)\mathrm{d}S+\int_{\Omega} w\nabla^2 f\,\mathrm{d}V\,, \tag{2.2}$$

where $\partial/\partial n$ denotes the derivative in the outward normal direction.

Hence, if we require f to satisfy

$$\nabla^2 f=0\,, \tag{2.3}$$

then we have our basic integral result

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\int_{\partial\Omega}\left(f\,\frac{\partial w}{\partial n}-w\frac{\partial f}{\partial n}\right)\mathrm{d}S\,, \tag{2.4}$$

which holds for any harmonic $f(\mathbf{x})$.

We now consider boundary conditions on (1.3) in the form

$$\begin{aligned}
&w=g_1(\mathbf{x},t) && \text{on } \partial\Omega_1\,,\\
&\frac{\partial w}{\partial n}=g_2(\mathbf{x},t) && \text{on } \partial\Omega_2\,,\\
&\frac{\partial w}{\partial n}+K(\mathbf{x},t)w=g_3(\mathbf{x},t) && \text{on } \partial\Omega_3\,,
\end{aligned}$$

where K, g_1, g_2 and g_3 are given, $\partial\Omega_1$, $\partial\Omega_2$ and $\partial\Omega_3$ are non-intersecting and further $\partial\Omega_1\cup\partial\Omega_2\cup\partial\Omega_3=\partial\Omega$. If we impose conditions on (2.3) of the form

$$\begin{aligned}
&f=0 && \text{on } \partial\Omega_1\,,\\
&\frac{\partial f}{\partial n}=0 && \text{on } \partial\Omega_2\,,\\
&\frac{\partial f}{\partial n}+K(\mathbf{x},t)f=0 && \text{on } \partial\Omega_3\,,
\end{aligned} \tag{2.5}$$

then the value of the right-hand side of (2.4) may be calculated, independently of $c(\mathbf{x},t)$. Specifically, we have

$$\frac{\mathrm{d}I}{\mathrm{d}t}=-\int_{\partial\Omega_1} g_1\,\frac{\mathrm{d}f}{\mathrm{d}n}\,\mathrm{d}S+\int_{\partial\Omega_2} fg_2\,\mathrm{d}S+\int_{\partial\Omega_3} fg_3\,\mathrm{d}S\,. \tag{2.6}$$

If $g_1=g_2=g_3=0$, then I is an integral invariant; in the general case its value may be determined over all time from (2.6), provided that the integral (2.1) exists for all time.

If Ω is a finite domain then the only bounded solution of (2.3) and (2.5) is usually $f \equiv 0$; the important exception is when $\partial\Omega_2 = \partial\Omega$ so that $f = 1$ is a solution and (2.6) gives the usual conservation of total mass result for the problem in which the flux $\partial w/\partial n$ is given everywhere on the boundary. However, unbounded solutions for f can also give integral results. As a simple example we consider the two-dimensional case when

$$\Omega = \{(r, \theta): 0 < r < 1, 0 < \theta < \theta_0\} ,$$

where θ_0 is a constant, and $\partial\Omega_1 = \partial\Omega$. Possible forms for f are then

$$f = [r^{-n\pi/\theta_0} - r^{n\pi/\theta_0}] \sin \frac{n\pi\theta}{\theta_0} ,$$

with $n = 1, 2, 3, \ldots$, which for $n = 1$, $\theta_0 > \pi/2$ (at least) gives bounded integrals

$$\int_\Omega fc \, \mathrm{d}V \equiv \int_0^{\theta_0} \int_0^1 fcr \, \mathrm{d}r \, \mathrm{d}\theta .$$

We note, however, that since f is singular at $r = 0$, care must be exercised in taking the limit $r \to 0$ in both the surface and the volume integral. In practice, an expression for the local behaviour at $r = 0$ in terms of the global behaviour results.

If, however, Ω is an infinite domain the position is rather different. If we suppose that $c \to c_0$ (a constant) at infinity then we should consider integrals of the form

$$I[f] = \int_\Omega f(\mathbf{x})(c - c_0) \, \mathrm{d}V , \tag{2.7}$$

and the only constraint on f at infinity is that it does not grow too fast so that the integral (2.7) is convergent. If $D(c) \to D_0 > 0$ as $c \to c_0$, with D_0 constant, then w and $c - c_0$ are likely to behave like $\exp(-r^2/4D_0 t)$ as $r = |\mathbf{x}| \to \infty$ so that this requirement on f is not a stringent one. Because of the weakness of the requirement on the behaviour of f at infinity there may be many forms which are bounded for finite $|\mathbf{x}|$ and which lead to finite integrals of the form (2.7).

Another context in which boundary conditions on f may not need be applied around the whole of $\partial\Omega$ is when we have a moving-boundary problem, so that $\Omega \equiv \Omega(t)$ and $\partial\Omega(t)$ must be determined as part of the solution. These problems arise frequently, especially when $D \to 0$ as $c \to 0$ (or, to be more accurate, when $\int_0^c D(c)c^{-1} \, \mathrm{d}c < \infty$) since then if c initially has compact support then it does so for all time, with $c = 0$ on the moving boundary. When Ω varies with time, (2.2) becomes

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \int_{\partial\Omega} \left(f\left(\frac{\partial w}{\partial n} + q_n c \right) - w \frac{\partial f}{\partial n} \right) \mathrm{d}S + \int_\Omega w \nabla^2 f \, \mathrm{d}V , \tag{2.8}$$

where $q_n(\mathbf{x}, t)$ is the outward normal velocity of the moving boundary $\partial\Omega(t)$. Since $\partial\Omega(t)$ is to be determined two boundary conditions must be given there. If mass is conserved at the moving boundary we require

$$\frac{\partial w}{\partial n} + q_n c = 0 \quad \text{on } \partial\Omega(t)$$

and taking $c_0 = 0$ in (1.2) and assuming that $c = w = 0$ on (and outside) $\partial\Omega(t)$, we then have

$$\int_{\partial\Omega} \left(f\left(\frac{\partial w}{\partial n} + q_n c \right) - w \frac{\partial f}{\partial n} \right) \mathrm{d}V = 0 ,$$

at least if f and $\partial f/\partial n$ are bounded on $\partial\Omega(t)$. Hence as long as (2.3) is satisfied I is in general an integral invariant of the problem, without imposing boundary conditions on f.

If mass is not conserved on the moving boundary, for example for Stefan problems where typically

$$\frac{\partial w}{\partial n} + q_n c = -Lq_n$$

holds on $\partial\Omega$ (where L is a given constant), then integral invariants are not usually available since the position of $\partial\Omega(t)$ is not known without solving the whole problem.

For problems in which only part of the boundary is moving, the contributions from the fixed parts of the boundary to the surface integral in (2.8) must be retained.

We conclude this section by making a few points about our results, including their relation to earlier work.

(1) In one dimension, taking $\Omega = \{x: 0 < x < s(t)\}$, (2.8) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{s(t)} f(x) c \, \mathrm{d}x = \left(f\left(\frac{\partial w}{\partial x} + \frac{\mathrm{d}s}{\mathrm{d}t} c \right) - w \frac{\partial f}{\partial x} \right)\bigg|_{x=s} - \left(f \frac{\partial w}{\partial x} - w \frac{\partial f}{\partial x} \right)\bigg|_{x=0} + \int_0^s w \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \, \mathrm{d}x .$$

Assuming that $c = w = \partial w/\partial x = 0$ on the moving boundary $x = s(t)$ (or as $x \to +\infty$ if $s = \infty$) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^s f(x) c \, \mathrm{d}x = \left(w \frac{\mathrm{d}f}{\mathrm{d}x} - f \frac{\mathrm{d}w}{\partial x} \right)\bigg|_{x=0} + \int_0^s w \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \, \mathrm{d}x .$$

Satisfying $\mathrm{d}^2 f/\mathrm{d}x^2 = 0$ we obtain the conservation of mass result

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^s c \, \mathrm{d}x = -\frac{\partial w}{\partial x}\bigg|_{x=0} ,$$

and the centre of mass result

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^s xc \, \mathrm{d}x = w\bigg|_{x=0} ,$$

which are well known (see, for example, Barenblatt [4] and Vazquez [23]).

Choosing $f(x) = x^n$, $n = 2, 3, 4, \ldots$ gives the other moment results:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^s x^n c \, \mathrm{d}x = n(n-1) \int_0^s x^{n-2} w \, \mathrm{d}x .$$

This result (and similar results in higher dimensions) has been used to construct approximate solutions to nonlinear diffusion problems by assuming a particular form for c and exactly satisfying a finite number of moment relations; see Andriankin [1] and Pomraning [17–19]. Similar results have also been applied to linear convection-diffusion problems (see Aris [2]).

(2) For linear diffusion ($D(c) = D_0$ with D_0 constant) we can generalise our results by writing

$$I[f] = \int_\Omega f(\mathbf{x}, t) c \, dV$$

to give (for fixed Ω)

$$\frac{dI}{dt} = D_0 \int_{\partial\Omega} \left(f \frac{\partial c}{\partial n} - c \frac{\partial f}{\partial n} \right) dS + \int_\Omega \left(\frac{\partial f}{\partial t} + D_0 \nabla^2 f \right) c \, dV ,$$

which reduces to a surface integral if f is chosen to satisfy the backward heat equation

$$\frac{\partial f}{\partial t} + D_0 \nabla^2 f = 0 .$$

By determining all such f satisfying the appropriate boundary conditions, it is in principle possible to essentially completely describe the diffusion process.

The corresponding one-dimensional result was given by Steinberg and Wolf [22], and moment methods were applied to the analysis of one-dimensional diffused profiles by Ghez et al. [9], who assumed that the diffusion was linear.

(3) The simplest solutions to (2.3) are $f = 1$ and $f = x$, y or z giving the conservation of mass result

$$\frac{d}{dt} \int_\Omega c \, dV = \int_{\partial\Omega} \frac{\partial w}{\partial n} \, dS ,$$

and the centre of mass result

$$\frac{d}{dt} \int_\Omega \mathbf{x} c \, dV = - \int_{\partial\Omega} w \hat{\mathbf{n}} \, dS ,$$

where $\hat{\mathbf{n}}$ is the unit outward normal. These multi-dimensional results were given in King [13].

(4) For radial symmetry in N dimensions, so that

$$\frac{\partial c}{\partial t} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial w}{\partial r} \right) ,$$

the most general solutions to (2.3) are

$$f = \alpha + \beta r^{2-N} \qquad N \neq 2 ,$$

$$f = \alpha + \beta \ln r \qquad N = 2 ,$$

where α and β are arbitrary constants. For $\alpha = 1$, $\beta = 0$ we have the conservation of mass result

$$\frac{d}{dt} \int_{r_0}^{r_1} r^{N-1} c \, dr = \left[r^{N-1} \frac{\partial w}{\partial r} \right]_{r_0}^{r_1} \tag{2.9}$$

while taking $\alpha = 0$, $\beta = 1$ gives a generalisation of the one-dimensional centre of mass result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{r_0}^{r_1} rc\,\mathrm{d}r = \left[r\frac{\partial w}{\partial r} + (N-2)w\right]_{r_0}^{r_1} \tag{2.10}$$

(which is the same as (2.9) when $N = 2$), and for $N = 2$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{r_0}^{r_1} r\ln rc\,\mathrm{d}r = \left[r\ln r\frac{\partial w}{\partial r} - w\right]_{r_0}^{r_1}, \tag{2.11}$$

with r_0 and r_1 constant in each case.

(5) For the 'mesa' problem (Elliott et al. [8]) in which $D(c) = c^m$ with $m \to \infty$ we have

$$c \sim t^{-1/m} \quad \text{in } \Omega_m(t) \quad \text{as } m \to \infty, \tag{2.12}$$

where $\Omega_m(t) \subset \mathbb{R}^N$ is the mesa region in which c is uniform at leading order, while

$$c \sim C(\mathbf{x}) \quad \text{in } \mathbb{R}^N - \Omega_m, \tag{2.13}$$

where $c = C(\mathbf{x})$ at $t = 0$. Then if

$$I[f] = \int_{\mathbb{R}^N} f(\mathbf{x})c\,\mathrm{d}V,$$

we have $\mathrm{d}I/\mathrm{d}t = 0$ if $\nabla^2 f = 0$ (assuming suitable behaviour at infinity), so that

$$I = \int_{\mathbb{R}^N} f(\mathbf{x})C(\mathbf{x})\,\mathrm{d}V$$

for all time. This implies that

$$\int_{\Omega_m} f(\mathbf{x})\,\mathrm{d}V \sim t^{1/m}\int_{\Omega_m} f(\mathbf{x})C(x)\,\mathrm{d}V. \tag{2.14}$$

In particular in two dimensions we can choose

$$f(\mathbf{x}) = z^n, \quad \text{with } n = 0, 1, 2, \ldots \quad \text{and} \quad z = x + \mathrm{i}y,$$

to give the corresponding moment results. The mesa problem is closely related to the Hele–Shaw problem for which it is known (Richardson [20, 21]; see also Howison [12]) that the motion is essentially completely described by the moments and an analogous result presumably holds here. It was pointed out in Elliott et al. [8] and King [13] that in one dimension the leading-order solution to the mesa problem is completely determined by conservation of mass and centre of mass, and our result essentially generalises this to two (and probably more) dimensions.

For other diffusion problems there is far more information contained in $c(\mathbf{x}, t)$ than that expressed by the asymptotic results (2.12) and (2.13), and our integral invariants certainly do not completely characterise the problem.

(6) Convenient forms to use for f are often, in two dimensions,

$$f = r^n \cos n\theta \quad \text{and} \quad f = r^n \sin n\theta ,$$

and in three dimensions

$$f = r^n P_n^m(\cos\theta) \cos m\varphi , \quad m = 0, 1, \ldots, n ,$$

and

$$f = r^n P_n^m(\cos\theta) \sin m\varphi , \quad m = 1, 2, \ldots, n ,$$

where $n = 0, 1, \ldots,$ and P_n^m denotes the associated Legendre function. We will not, however, discuss detailed applications of these here.

Terms of the form, in two dimensions:

$$f = \ln r, \qquad f = r^{-n} \cos n\theta \qquad \text{and} \quad f = r^{-n} \sin n\theta ,$$

and in three dimensions:

$$f = r^{-n} P_{n-1}^m(\cos\theta) \cos m\varphi , \quad m = 0, 1, \ldots, n-1 ,$$

and

$$f = r^{-n} P_{n-1}^m(\cos\theta) \sin m\varphi , \quad m = 1, 2, \ldots, n-1 ,$$

with $n = 1, 2, \ldots$ can also be used. Care, however, must be taken to ensure that the integral (2.1) exists, and the term

$$\int_\Omega w \nabla^2 f \, \mathrm{d}V$$

in (2.2) must be handled appropriately since these forms of f are not harmonic at $r = 0$.

(7) Although the original problem is nonlinear, (2.2) is linear in f so linear combinations of integral invariants also give invariants.

(8) In King [15] similarity solutions which correspond to the invariants (2.9)–(2.10) are explicitly constructed for power-law diffusivities.

3. Inhomogeneous diffusion

Grundy [10] has obtained two integral invariants for inhomogeneous nonlinear diffusion in one dimension which we now generalise by considering equations of the form:

$$\rho(\mathbf{x}) \frac{\partial c}{\partial t} = \nabla \cdot (K(\mathbf{x}) \nabla w) , \tag{3.1}$$

where ρ and K are prescribed and w is related to c by (1.2).

Writing

$$I[f]=\int_\Omega f(\mathbf{x})\rho c\,\mathrm{d}V\,,$$

then if Ω is fixed we have

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\int_{\partial\Omega} K\Big(f\,\frac{\partial w}{\partial n}-w\,\frac{\partial f}{\partial n}\Big)\,\mathrm{d}S+\int_\Omega w\nabla\cdot(K\nabla f)\,\mathrm{d}V\,,$$

so that integral results are obtained from solutions to the linear equation

$$\nabla\cdot(K\nabla f)=0\,. \tag{3.2}$$

Once again $f=1$ gives conservation of mass, while in one dimension (3.2) also gives

$$f=\int_0^x \frac{1}{K(x')}\,\mathrm{d}x'\,,$$

which gives the generalised centre of mass result obtained by Grundy [10].

When (3.1) is linear, generalisations are again made possible by writing $f(\mathbf{x}, t)$. We also note that radially symmetric homogeneous problems are equivalent to particular inhomogeneous one-dimensional problems.

4. Higher-order diffusion

Recently (Smyth and Hill [23], Bernis and Friedman [6]) interest has arisen in higher-order nonlinear diffusion equations of the form

$$\frac{\partial u}{\partial t}=-\nabla\cdot(F(u)\nabla(\nabla^2 u)) \tag{4.1}$$

and in even-higher-order problems (the equation

$$\frac{\partial u}{\partial t}=\nabla\cdot(u^3\nabla(\nabla^4 u))$$

has arisen in the modelling of semiconductor-device fabrication (King [14])).

Equation (4.1) has the conservation of mass result

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\,\mathrm{d}V=-\int_{\partial\Omega} F(u)\,\frac{\partial}{\partial n}\,(\nabla^2 u)\,\mathrm{d}S\,,$$

but in general there are apparently no other integral results of the forms we have been discussing.

In the linear case $F(u)=1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f(\mathbf{x},t)u\,\mathrm{d}V=\int_{\partial\Omega}\Big(\frac{\partial}{\partial n}\,(\nabla^2 f)u-\nabla^2 f\,\frac{\partial u}{\partial n}+\frac{\partial f}{\partial n}\,\nabla^2 u-f\,\frac{\partial}{\partial n}\,(\nabla^2 u)\Big)\,\mathrm{d}S$$
$$+\int_\Omega u\Big(\frac{\partial f}{\partial t}-\nabla^4 f\Big)\,\mathrm{d}V\,,$$

so that integral results can be constructed from solutions to the backward equation

$$\frac{\partial f}{\partial t} = \nabla^4 f .$$

Particularly simple cases are represented by $f = 1, x, x^2$ and x^3.

Another case in which a further integral result is available is the one-dimensional problem with $F(u) = u + u_0$, with u_0 constant. Then we have the centre of mass result

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_0}^{x_1} xu \,\mathrm{d}x = -\left[x(u + u_0) \frac{\partial^3 u}{\partial x^3} - (u + u_0) \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 \right]_{x_0}^{x_1}$$

if x_0 and x_1 are constant. If $u_0 = 0$, so that

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} \left(u \frac{\partial^3 u}{\partial x^3} \right), \tag{4.2}$$

a mass-preserving similarity solution can be written down explicitly (Smyth and Hill [23]). If we seek a solution to (4.2) which conserves the first moment

$$\int_0^\infty xu \,\mathrm{d}x ,$$

we write $u = t^{-1/3} f(x/t^{1/6})$ and we are able to integrate the corresponding ordinary differential equation once to give

$$\alpha + \frac{1}{6} \eta^2 f = \eta f \frac{\mathrm{d}^3 f}{\mathrm{d}\eta^3} - f \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + \frac{1}{2} \left(\frac{\mathrm{d}f}{\mathrm{d}\eta} \right)^2 ,$$

where α is an arbitrary constant.

We may generalise the Smyth and Hill [23] solution to higher dimensions by seeking solutions to the radially symmetric equation

$$\frac{\partial u}{\partial t} = -\frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} u \frac{\partial}{\partial r} \left(\frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial u}{\partial r} \right) \right) \right)$$

of the form

$$u = t^{-N/(N+4)} f(r/t^{1/(N+4)}) ,$$

in order to conserve the total mass

$$\int_0^\infty r^{N-1} u \,\mathrm{d}r .$$

Setting the first constant of integration to zero we may then obtain:

$$f = \frac{1}{8(N+2)(N+4)} (\alpha + \beta \eta^{2-N} + \gamma \eta^2 + \eta^4) , \qquad N \neq 2 ,$$

$$f = \frac{1}{192} (\alpha + \beta \ln \eta + \gamma \eta^2 + \eta^4) , \qquad N = 2 ,$$

where α, β and γ are arbitrary constants.

Similar results can be obtained for equations of even higher order than (4.1).

5. A radially symmetric example

Here we consider a simple example which illustrates the application of some of our results.

We consider

$$\frac{\partial c}{\partial t} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1} \frac{\partial w}{\partial r}\right),$$

$$\begin{aligned} &\text{at } r = 1 && w = 0\,, \\ &\text{as } r \to \infty && w \to 0\,, \\ &\text{at } t = 0 && c = C(r)\,, \end{aligned} \tag{5.1}$$

where the initial condition $C(r)$ is specified. We illustrate how the large-time behaviour of (5.1) may be determined when $D(c) = c^n$ so that

$$w = \frac{1}{n+1} c^{n+1}\,, \tag{5.2}$$

choosing $c_0 = 0$; we note that $n > -1$ is needed. Our large-time results apply also to more general problems with $D(c) \sim c^n$ as $c \to 0$.

From (2.9) to (2.11) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_1^\infty r^{N-1} c \,\mathrm{d}r = -\left.\frac{\partial w}{\partial r}\right|_{r=1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_1^\infty rc \,\mathrm{d}r = -\left.\frac{\partial w}{\partial r}\right|_{r=1},$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_1^\infty r \ln rc \,\mathrm{d}r = 0 \quad \text{for } N = 2\,,$$

assuming in each case that $w \to 0$ sufficiently fast as $r \to \infty$. Hence we have integral invariants

$$\int_1^\infty r(r^{N-1} - 1)c \,\mathrm{d}r = \int_1^\infty r(r^{N-1} - 1)C \,\mathrm{d}r\,, \qquad N \neq 2\,, \tag{5.3}$$

and

$$\int_1^\infty r \ln rc \,\mathrm{d}r = \int_1^\infty r \ln rC \,\mathrm{d}r\,, \qquad N = 2\,. \tag{5.4}$$

From (5.3) we see that if $N = 1$

$$\int_1^\infty (x-1)c \,\mathrm{d}x = \int_1^\infty (x-1)C \,\mathrm{d}x\,,$$

writing x in place of r. This is the usual centre of mass result (if we translate x by 1) and the large-time solution corresponding to (5.2) with $n > -1$ is given by the dipole solution of Barenblatt and Zel'dovich [5]. This dipole solution has been generalised to higher dimensions by King [15], but we shall see that this generalisation is not appropriate to describing the large-time behaviour of (5.1) when $N = 2$ or 3.

When $N = 3$, (5.3) becomes

$$\int_1^\infty r(r-1)c \, \mathrm{d}r = \int_1^\infty r(r-1)C \, \mathrm{d}r \, . \tag{5.5}$$

Since we expect the concentration profile to spread out more and more with increasing time we expect the large-time behaviour of the left-hand side of (5.5) to be dominated by the total-mass term

$$\int_0^\infty r^2 c \, \mathrm{d}r \, ,$$

the diffusion lengthscale becoming much greater than 1. The appropriate large-time behaviour for this constraint for (5.2) with $n > -\frac{2}{3}$ is given by the instantaneous source solution of Barenblatt [3] and Pattle [16] which takes the form

$$c = t^{-3/(3n+2)} f[r/t^{1/(3n+2)}] \tag{5.6}$$

so that

$$-\frac{1}{3n+2} \eta f = f^n \frac{\mathrm{d}f}{\mathrm{d}\eta} \, , \tag{5.7}$$

and the constant of integration for (5.7) is determined explicitly from the initial data using (5.5). This solution does not satisfy the boundary condition $w = 0$ on $r = 1$, but corresponds to the leading-order outer solution of a singular perturbation problem in which the outer scaling is given by

$$r = \mathrm{O}[t^{1/(3n+2)}]$$

for large t. The inner scaling is $r = \mathrm{O}(1)$, so that at leading order

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial w}{\partial r} \right) = 0 \, ,$$

$$\text{at } r = 1: \qquad w = 0 \, ,$$

$$\text{as } r \to +\infty: \qquad w \to t^{-3(n+1)/(3n+2)} \frac{f^{n+1}(0)}{n+1} \, ,$$

matching with (5.6), and the leading-order inner solution is

$$w = t^{-3(n+1)/(3n+2)} \frac{f^{n+1}(0)}{n+1} \left(1 - \frac{1}{r}\right) .$$

We note that the generalised dipole solution given in King [15] corresponds to the invariance of

$$\int_0^\infty rc\,\mathrm{d}r$$

and is therefore not appropriate here.

The intermediate case $N = 2$ is more delicate because of the occurrence of $\ln r$ terms. We again expect the diffusion length to increase with time and we assume an outer scaling of $r = O(s(t))$, where $s(t)$ is to be determined but where $s(t) \to \infty$ at $t \to \infty$. Introducing $\eta = r/s(t)$ the left-hand side of (5.4) is dominated by

$$s^2 \ln s \int_0^\infty \eta c\,\mathrm{d}\eta$$

which gives a conservation of mass type condition and implies that

$$c \sim s^{-2} \ln^{-1} s\, f(r/s) \tag{5.8}$$

at $t \to \infty$. Substituting (5.8) into

$$\frac{\partial c}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rc^n \frac{\partial c}{\partial r}\right) \tag{5.9}$$

to obtain a balance we find that we need

$$s^{2n+1} \ln^n s \frac{\mathrm{d}s}{\mathrm{d}t} = O(1)$$

which implies that the relevant diffusion lengthscale is

$$s(t) = t^{1/2(n+1)} \ln^{-n/2(n+1)} t\,,$$

and for large time

$$c \sim t^{-1/(n+1)} \ln^{-1/(n+1)} t f(r/t^{1/2(n+1)} \ln^{-n/2(n+1)} t)\,. \tag{5.10}$$

If the $\ln t$ terms were absent (5.10) would represent the usual (Barenblatt–Pattle) instantaneous source solution to (5.9); in fact (5.10) is not the exact form of any similarity solution of (5.9) but a large time balance shows that f satisfies the same ordinary differential equation as the instantaneous source solution, namely

$$-\frac{1}{2(n+1)}\left(2\eta f + \eta^2 \frac{\mathrm{d}f}{\mathrm{d}\eta}\right) = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f^n \frac{\mathrm{d}f}{\mathrm{d}\eta}\right). \tag{5.11}$$

Integrating (5.11) gives

$$-\frac{1}{2(n+1)}\eta^2 f = \eta f^n \frac{\mathrm{d}f}{\mathrm{d}\eta}$$

and a constraint determined from (5.4)

$$\int_0^\infty \eta f(\eta)\,\mathrm{d}\eta = 2(n+1)\int_1^\infty r \ln rC(r)\,\mathrm{d}r\,,$$

determines the final constant of integration. The condition $n > -1$ is again needed for a solution of the required form to exist. We note that the total mass associated with (5.10) decays in time as $\ln^{-1} t$.

Once again the outer solution (5.10) will not satisfy the condition $w = 0$ on $r = 1$ and in the inner region $r = O(1)$ the dominant balance is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = 0\,,$$

$$\text{at } r = 1: \quad w = 0\,,$$

with solution

$$w = \alpha(t) \ln r\,,$$

and matching at leading order gives

$$\alpha(t) = \frac{2f^{n+1}(0)}{t\ln^2 t}$$

(the matching is rather delicate but we omit details because it is outside the theme of this paper). We note, however, that (as in the three-dimensional case) the leading-order outer solution can be determined by using the integral invariant without any reference to the matching. Here the form of the large-time behaviour (5.10) would be extremely hard to determine without using the integral result.

For linear diffusion $n = 0$, and $N = 3$ we note that an exact solution to (5.1) for appropriate initial data is given by

$$c = \frac{A}{(t+t_0)^{3/2}}\left(1 - \frac{1}{r}\right) \mathrm{e}^{-(r-1)^2/4(t+t_0)}\,,$$

where A and t_0 are arbitrary constants. This is consistent with our asymptotic results.

It was noted in Hill [11] and King [15] that the instantaneous source and generalised dipole solutions were identical when $N = 2$. For linear diffusion we can generate a new solution for $N = 2$ by using the superposition principle. We consider

$$c = \lim_{N\to 2}\left\{A\left(t^{-N/2}\,\mathrm{e}^{-r^2/4t} - t^{(N/2)-2} r^{2-N}\,\mathrm{e}^{-r^2/4t}\right)\Big/(N-2)\right\}$$

which is essentially the difference between the two solutions and gives

$$c = At^{-1}\ln\left(\frac{r}{t}\right)\mathrm{e}^{-r^2/4t}$$

as an exact solution for $N = 2$. However, this solution is not appropriate to (5.1), and does not correspond directly to making the quantity

$$\int_0^\infty r \ln rc \, \mathrm{d}r \tag{5.12}$$

constant in time even though (5.12) can be derived as the limit of the difference between

$$\int_0^\infty r^{N-1} c \, \mathrm{d}r \quad \text{and} \quad \int_0^\infty rc \, \mathrm{d}r \,,$$

which are constant for the instantaneous source and dipole solutions respectively.

For linear diffusion the transformation

$$c^* = rc \,, \qquad r^* = r$$

converts the $N = 3$ problem into the $N = 1$ problem and maps the corresponding form of (2.9) into (2.10) and vice-versa. Similarly, for $n = -1$ the transformation

$$c^* = r^2 c \,, \qquad r^* = \ln r$$

maps the $N = 2$ problem into the $N = 1$ problem (as noted in King [15]) and maps (2.10) into (2.9) and (2.11) into (2.10). The following form of similarity variables for $N = 2$, $n = -1$, noted in King [15]:

$$c = r^2 \, \mathrm{e}^{-2\lambda t} f(\ln r / \mathrm{e}^{\lambda t}) \,,$$

where λ is an arbitrary constant, is appropriate for making (5.12) constant in time.

6. Discussion

We have derived a class of integral results for nonlinear diffusion equations and illustrated their application through a simple example. They are of particular value in determining the large-time behaviour of specific initial-boundary value problems as illustrated in Section 5; the methods could equally well be applied to more general problems such as when w is prescribed as a function of time on $r = 1$.

We have not discussed in detail the application to genuinely multi-dimensional problems here, but applications to nonlinear diffusion under a mask edge (an important process in semiconductor device fabrication) will be presented elsewhere.

References

1. E.I. Andriankin, Propagation of a non-self-similar thermal wave. *Sov. Phys. JETP* 35 (1959) 295–298.
2. R. Aris, On the dispersion of a solute in a fluid flowing through a tube. *Proc. Roy. Soc. Lond.* A 235 (1956) 67–77.
3. G.I. Barenblatt, On some unsteady motions of a liquid or a gas in a porous medium. *Prikl. Mat. i. Mekh.* 16 (1952) 67–78.
4. G.I. Barenblatt, On the approximate solution of problems of uniform unsteady filtration in a porous medium. *Prikl. Mat. i. Mekh.* 18 (1954) 351–370.
5. G.I. Barenblatt and Ia.B. Zel'dovich, On dipole-type solution in problems of unsteady filtration of gas in a polytropic system. *Prikl. Mat. i. Mekh.* 21 (1957) 718–720.

6. F. Bernis and A. Friedman, Higher order nonlinear degenerate parabolic equations. *J. Diff. Equations* 83 (1990) 179–206.
7. J. Crank, *The Mathematics of Diffusion*. Second Edition, Clarendon, Oxford (1975).
8. C.M. Elliott, M.A. Herrero, J.R. King and J.R. Ockendon, The mesa problem: diffusion patterns for $u_t = \nabla \cdot (u^m \nabla u)$ as $m \to \infty$. *IMA J. Appl. Math.* 37 (1986) 147–154.
9. R. Ghez, J.D. Fehribach and G.S. Oehrlein, The analysis of diffusion data by a method of moments. *J. Electrochem. Soc.* 132 (1985) 2759–2761.
10. R.E. Grundy, Large time solution of an inhomogeneous nonlinear diffusion equation. *Proc. R. Soc. Lond.* A 386 (1983) 347–372.
11. J.M. Hill, Similarity solutions for nonlinear diffusion – a new integration procedure. *J. Eng. Math.* 23 (1989) 141–155.
12. S.D. Howison, Bubble growth in porous media and Hele–Shaw cells. *Proc. Roy. Soc. Edin.* 102A (1986) 141–148.
13. J.R. King, D. Phil. thesis, Oxford University (1986).
14. J.R. King, The isolation oxidation of silicon: the reaction-controlled case. *SIAM J. Appl. Math.* 49 (1989) 1064–1080.
15. J.R. King, Exact similarity solutions to some nonlinear diffusion equations. *J. Phys. A.* 23 (1990) 3681–3697.
16. R.E. Pattle, Diffusion from an instantaneous point source with a concentration-dependent coefficient. *Q. J. Mech. Appl. Math.* 12 (1959) 407–409.
17. G.C. Pomraning, A moments method for describing the diffusion of radiation from a cavity. *J. Appl. Phys.* 38 (1967) 3845–3850.
18. G.C. Pomraning, A moments method for describing the diffusion of radiation from a cavity, pt. II. *J. Appl. Phys.* 39 (1968) 1479–1484.
19. G.C. Pomraning, Propagation of a nonspherical thermal wave. *J. Appl. Phys.* 43 (1972) 2722–2727.
20. S. Richardson, Hele–Shaw flows with a free boundary produced by the injection of fluid into a narrow channel. *J. Fluid Mech.* 56 (1972) 609–618.
21. S. Richardson, Some Hele–Shaw flows with time-dependent free boundaries. *J. Fluid Mech.* 102 (1981) 263–278.
22. S. Steinberg and K.B. Wolf, Symmetry, conserved quantities and moments in diffusive equations. *J. Math. Anal. Appl.* 80 (1981) 36–45.
23. N.F. Smyth and J.M. Hill, High-order nonlinear diffusion. *IMA J. Appl. Math.* 40 (1988) 73–86.
24. J.L. Vazquez, Asymptotic behaviour and propagation properties of the one-dimensional flow of gas in a porous medium. *Trans. Am. Math. Soc.* 277 (1983) 507–527.